HW PROBLEMS SET 2: ARGUMENT BY CONTRADICTION, INDUCTION, PIGEONHOLE PRINCIPLE

1. The union of nine planar surfaces, each of area equal to 1, has a total area equal to 5. Prove that the overlap of some two of these surfaces has an area greater than or equal to $\frac{1}{9}$.

2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

3. Prove that for any positive integer n there exists an n-digit number divisible by 2^n and containing only the digits 2 and 3.

4. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\{\frac{a_{n-1}}{a_n} : n \geq 1\}.$$

5. Show that every positive integer can be written as a sum of distinct terms of the Fibonacci sequence.

6. Let p be a prime number and a, b, c be integers such that a and b are not divisible by p. Prove that the equation $ax^2 + by^2 = c(\text{mod } p)$ has integer solutions.

7. In each square of a board 10×10, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.